MATH 320 NOTES, WEEK 3

Recall: $Span(S)$ is the set of all linear combinations of vectors in S.

Examples:

(1) $Span(\{\langle 1,0,0\rangle, \langle 0,1,0\rangle\}) = \{\langle a,b,0\rangle \in \mathbb{R}^3 \mid a,b \in \mathbb{R}\}$;
(2) $Span(\{\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 5,11,0\rangle\}) = \{\langle a,b,0\rangle \in \mathbb{R}^3 \mid a,b \in \mathbb{R}\}$;

Note that $Span(\{\langle 1,0,0\rangle, \langle 0,1,0\rangle\}) = Span(\{\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 5,11,0\rangle\})$, since $\langle 5,11,0\rangle \in Span(\{\langle 1,0,0\rangle, \langle 0,1,0\rangle\})$. I.e. the vectors $\{\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 5,11,0\rangle\}$ are linearly dependent.

Section 1.5 Linear Dependence and Independence.

Let V be a vector space over a field F and $S \subset V$.

Definition 1. *S is* **linearly dependent** *iff there are vectors $v_1, ..., v_n$ in S and scalars $a_1, ..., a_n$ not all zero such that*

$$a_1v_1 + a_2v_2 + ... + a_nv_n = 0.$$

Definition 2. *S is* **linearly independent** *iff whenever*

$$a_1v_1 + a_2v_2 + ... + a_nv_n = 0$$

for vectors $v_1, ..., v_n$ in S and scalars $a_1, ..., a_n$, then $a_1 = a_2 = ...a_n = 0$.

Note that the empty set $\emptyset$ is linearly independent.

Examples:

(1) In $\mathbb{R}^3$, the set $\{(1,1,1), (2,-1,0), (4,1,2)\}$ is linearly dependent since
$$2(1,1,1) + (2,-1,0) - (4,1,2) = (0,0,0).$$
(2) In $\mathbb{R}^3$, the set $\{(1,0,0), (0,1,1), (0,1,2)\}$ is linearly independent.
(3) In $P(F)$, the set $\{1, x, x^2, ..., x^n, ...\}$ is linearly independent.
(4) In F^n, the set $\{e_1, ..., e_n\}$ is linearly independent, and in $M_{k,n}(F)$, the set $\{E_{ij} \mid 1 \leq i \leq k, 1 \leq j \leq n\}$ is linearly independent.

Problem. Prove that in $P_2(F)$, the set $\{1+2x+x^2, 1+x, 1+x^2\}$ is linearly independent.

Proof. Suppose that $a(1+2x+x^2)+b(1+x)+c(1+x^2) = \vec{0}$. Then, regrouping the terms on the left hand side, we get $(a+b+c)+(2a+b)x+(a+c)x^2 = \vec{0}$. So,

- $a + b + c = 0$,
- $2a + b = 0$,

- $a + c = 0$

Solving this system, we get that $a = b = c = 0$.

□

Remark 3. Let V be a vector space and $S_1 \subset S_2 \subset V$.

(1) If S_1 is linearly dependent, then S_2 is linearly dependent.
(2) If S_2 is linearly independent, then S_1 is linearly independent.

Also note that:

(1) If $x \neq \vec{0}$, then $\{x\}$ is linearly independent.
(2) $\{\vec{0}\}$ is linearly dependent, since $1\vec{0} = \vec{0}$.
(3) If $\vec{0} \in S$, then S is linearly dependent.

Now let us look at the cases of two vectors and then of three vectors:

Lemma 4. *Let x, y, z be all nonzero vectors. Then*

(1) *$\{x, y\}$ are linearly dependent iff they are multiples of each other. I.e. $x = cy$ for some scalar c. (The proof will be on the homework)*
(2) *$\{x, y, z\}$ are linearly dependent iff x and y are multiples of each other or $z = ax + by$ for some scalars a, b.*

Theorem 5. *Suppose that S is linearly independent. Then $S \cup \{v\}$ is linearly dependent iff $v \in span(S)$.*

Proof. For the first direction, suppose that $S \cup \{v\}$ is linearly dependent. Then there are vectors $v_1, ..., v_n \in S$ and scalars $a_0, a_1, ..., a_n$, not all zero, such that $a_0 v + a_1 v_1 + a_2 v_2 + ... a_n v_n = 0$.

First note that $a_0 \neq 0$. For otherwise, we would have that $a_0 v + a_1 v_1 + a_2 v_2 + ... + a_n v_n = 0$, with $a_1, ..., a_n$, not all zero but that is a contradiction with S linearly independent.

We have

$$a_0 v = -a_1 v_1 - a_2 v_2 + ... - a_n v_n,$$

and since $a_0 \neq 0$, then

$$v = -\frac{a_1}{a_0} v_1 - \frac{a_2}{a_0} v_2 + ... - \frac{a_n}{a_0} v_n.$$

It follows that $v \in span(S)$.

For the other direction, suppose that $v \in span(S)$. Then for some scalars $a_1, ..., a_n$ and vectors $v_1, ..., v_n$ in S,

$$v = a_1 v_1 + ... + a_n v_n.$$

Then

$$1v - a_1 v_1 - ... - a_n v_n = 0,$$

and since the coefficient for v is 1, which is nonzero, this means that $\{v, v_1, ..., v_n\}$ are linearly dependent. Then $S \cup \{v\}$ is linearly dependent. □

Section 1.6 Bases and Dimension

Definition 6. *Suppose that $\beta \subset V$, for a vector space V. We say that β is a* **basis** *for V iff*

(1) *$Span(\beta) = V$,*
(2) *β is linearly independent.*

Examples:

(1) $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for $\mathbb{R}^3$.
(2) $\{(1,0,0),(0,1,1),(-2,0,1)\}$ is a basis for $\mathbb{R}^3$.
(3) More generally, $\{e_1,...,e_n\}$ is a basis for F^n.
(4) $\{1,x,x^2\}$ is a basis for $P_2(F)$.
(5) $\{1,x,x^2,...,x^n,...\}$ is a basis for $P(F)$.

Lemma 7. *$\beta = \{u_1,...,u_n\}$ is a basis for V iff each vector $x \in V$ can be uniquely expressed as a linear combinations of the vectors in β.*

Proof. For the first direction, suppose that β is a basis. Then by definition, each vector can be expressed as a linear combinations of the vectors in β. So, we just have to prove uniqueness. To that end, let $x \in V$ and suppose that for some scalars $a_1,...,a_n,b_1,...,b_n$,

- $x = a_1u_1 + ... + a_nu_n$, and
- $x = b_1u_1 + ... + b_nu_n$.

We have to show that $a_1 = b_1$, $a_2 = b_2$, ..., $a_n = b_n$. By subtracting the two equalities, we get that:

$$a_1u_1 + ... + a_nu_n - (b_1u_1 + ... + b_nu_n) = \vec{0}.$$

By distributing and reordering the terms, we get that

$$(a_1 - b_1)u_1 + (a_2 - b_2)u_2 + ... + (a_n - b_n)u_n = \vec{0}.$$

But then since β is linearly independent, we have

$$(a_1 - b_1) = (a_2 - b_2) = ... = (a_n - b_n) = 0.$$

So $a_1 = b_1$, $a_2 = b_2$, ..., $a_n = b_n$.

For the other direction, suppose that each vector $x \in V$ can be uniquely expressed as a linear combinations of the vectors in β. We have to show that β is a basis. We are already given by assumption that $Span(\beta) = V$. So we have to show that β is linearly independent. To that end, suppose that for some scalars $a_1,...,a_n$,

$$a_1u_1 + ... + a_nu_n = \vec{0}.$$

But we also have that $0u_1 + ... + 0u_n = \vec{0}$. Since $\vec{0}$ is uniquely expressed as a linear combination of vectors in β, it follows that $a_1 = 0, a_2 = 0, ..., a_n = 0$. □

Next we want to define **the dimension** of a vector space V. The dimension of V will be the size of a basis for V. But for this notion to be well defined, we need two things:

(1) each vector space has a basis, and
(2) if β, γ are two bases for V, then they have the same size.